Examinations and Verifications of the Heteroscedasticity in the SD Models with Lagged Variables by CU AMDAHL S. 470-V/8 Computer Machine Simulations

Tokumaru KIMURA

Introduction

First of all, I would like to show the fundamental system simulator expressions that I had used in performing computations and calculations by operating CU (IBM) AMDAHL S. 470-V/8 computer machine in Columbia University. (N.B.: The computer programming lists that had been printed out are omitted to show herewith in this paper for want of space. Because various routines for computations and calculations have 52,000 line sentences in total).

Of course, the following fifteen fundamental system simulator expressions in all are defined as the principal framework that logically supports various argumentations, examinations and verifications in this paper.

SSE (1) $y_t = x_t \triangle + u_t, \quad (t=1, \cdots\cdots, N).$

SSE (2) $(X'X)^{-1}X'\Sigma X(X'X)^{-1}.$

SSE (3) $y_t = \sum_{j=1}^{p} \triangle_j y_{t-j} + u_t.$

SSE (4) $y_t = Z_t' \triangle + u_t.$

SSE (5) $y = Z\triangle + u.$

SSE (6) $N^{-1}(Z'Z)^{-1}Z'\hat{\Sigma}Z(Z'Z)^{-1}.$

SSE (7) $y_t = \sum_{j=0}^{\infty} a_j u_{t-j}, \quad a_0 = 1, \quad \sum_{j=1}^{\infty} a_j^2 < \infty.$

SSE (8) $(\hat{\triangle} - \triangle) = (Z'Z)^{-1}Z'u = \left(\frac{1}{N}Z'Z\right)^{-1}\{(\sum_{t=1}^{N} Z_t u_t)/N\}.$

SSE (9) $(Z'Z)/N \xrightarrow{\text{a.s.}} V.$

SSE (10) $\frac{1}{N}\sum_{t=1}^{N} u_{t-r-j}u_t \xrightarrow{\text{a.s.}} 0$

SSE(11) $\{Z'(\hat{\Sigma}-\Sigma)Z\}/N = -\frac{2}{N}\sum_{t=1}^{N} Z_t Z_t'(\hat{\triangle}-\triangle)u_t Z_t' + \frac{1}{N}\sum_{t=1}^{N} Z_t(Z_t'(\hat{\triangle}-\triangle))^2 Z_t'.$

SSE(12) $-\frac{2}{N}\sum_{t=1}^{N} Z_t Z_t'(\hat{\triangle}-\triangle)u_t Z_t' \xrightarrow{\text{a.s.}} 0.$

SSE(13) $\frac{1}{N}\sum_{t=1}^{N} Z_t(Z_t'(\hat{\triangle}-\triangle))^2 Z_t' \xrightarrow{\text{a.s.}} 0.$

SSE(14) $\frac{1}{N}Z'(\hat{\Sigma}-\Sigma)Z \xrightarrow{\text{a.s.}} 0.$

SSE(15) $y_t = \triangle y_{t-1} + u_t.$

By the way, the purpose of this paper is to do examinations and verifications on various analytical results that White et al. had drawn out concerning with the Heteroscedasticity in models (i.e.—— System Dynamic Models (SD Models)) with lagged dependent and endogenous variables[1]. Especially, herewith in this paper, I would like to verify that some analytical results among them hold when exogenous variables are replaced by lagged endogenous variables[2]. Here, to simplify the proof that lagged values of y_t appear among the x_t, it is assumed that the above-shown system simulator expression (i.e.—— SSE (1)) could be defined by the implications that are explained by the SSE (3).

In their papers, White et al. considered a slightly more general specification than that above by allowing the exogenous regressors to be stochastic. They observed that their results generalize to allow some of the x_t to be endogenous variables, the appropriate estimator of $\triangle$ now being some form of instrumental variable estimator. These variations do not exhaust the possible situations faced by applied researchers, however, even if one concentrates only upon the classical regression model. In particular, the important case when some of the x_t are lagged values of y_t, is not to be covered; but it is very common for researchers to account for system dynamics by the use of such lagged variables[3].

Generally speaking, the regression model and its various extensions are perhaps the most widely used technical means in applied econometric and system dynamics theories. Because of the central role much effort has been devoted to the construction of diagnostic statistics that would reveal possible failures in the assumptions underlying it. One of these assumptions has been that the variances of the distributions and of distrubances be constant or homogeneous homoscedastic; a failure of this condition leads to invalid inferences whenever the traditional formula for the Ordinary Least Square (OLS) variance is utilized in the construction of "t statistics." For this reason, a number of proposals have been made that seek to eliminate any heteroscedasticity.

Such knowledge seems to be presumptuous. It is difficult enough to specify the regression part of the model, about which there is generally some theoretical guidance, without being required to state exactly how the error variances change. Therefore, the proposal by White et al. to make allowances for any form of heteroscedasticity by adjusting the OLS formula variance has considerable appeal.

Here, let me discuss the White's analytical result[4]. His analytical result may be formalized in the following way. Consider the regression-type SSE (1). In this SSE (1), the $1 * p$ nonstochastic vector x_t are exogenous and disturbances u_t are independently but not identically distributed with zero means and variances σ_t^2, i.e., the disturbances are heteroscedastic. If the $N * 1$ vector y is such that $y = X\triangle + u$, with X and u defined in an obvious way, the covariance matrix of the OLS estimator of $\triangle$, $\hat{\triangle} = (X'X)^{-1}X'y$, is to be the SSE (2). And furthermore, in this SSE (2), $\Sigma = E(uu') = \mathrm{diag}\{\sigma_t^2\}$.

In White's paper[5], he proposed that the SSE (2) be estimated by replacing Σ by $\hat{\Sigma} = \mathrm{diag}\{\hat{u}_t^2\}$ where $\hat{u}_t = y_t - x_t\hat{\triangle}$. White demonstrated that $\hat{\triangle} \xrightarrow{\text{a.s.}} \triangle$ and that the proposed estimator of the covariance matrix is strongly consistent.

By the way, in this paper, the followings consist of the four PARTS: THE PART II(METHOD) states the suppositions needed for the proof and also a LEMMA that is central to it. THE PART III(EXAMINATIONS AND VERIFICATIONS) gives examinations and verifications concerned about the strong consistency of the estimators in the system dynamics models. And the PART IV (SIMULATION EXPERIMENTS BY OPERATING CU(IBM) AMDAHL S. 470-V/8 COMPUTER MACHINE AND THEIR RESULTS) shows some limited system simulation experiments to gauge the success of the estimators. Finally, the PART V (CONCLUSION) provides some concluding remarks.

Method

First of all, for the SSE (3), I can make the following suppositions:

Supposition (a)— —All the zeros of $1-\sum_{j=1}^{p}\Delta_j Z^j=0$ lie outside the unit circle.

Supposition (b)——If $Z_t'=(y_{t-1},\cdots\cdots,y_{t-p})$, $\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}E(Z_tZ_t')=V>0$.

Supposition (c)——The u_t, $t=1,2,\cdots\cdots$, are martingale differences with $E(u_su_t)=0$, $s\neq t$, and $E(u_su_t)=\sigma_t^2<\sigma^2<\infty$, $s=t$.

Supposition (d)——There exist positive constants δ, D such that

$$E(|u_iu_ju_ku_l|^{1+\delta})\leq D<\infty,$$
$$(i,j,k,l=1,2,\cdots\cdots).$$

With respect to Supposition (b), Suppositions (c) and (a) ensure that $\lim_{N\to\infty} N^{-1}\sum_{t=1}^{N}E(Z_tZ_t')$ is bounded. The exact necessary and sufficient set of conditions needed to ensure nonsingularity are difficult to specify precisely. When $p=1$, the condition is $\lim_{N\to\infty} N^{-1}\sum_{t=1}^{N}\sum_{j=0}^{\infty}\Delta_1^{2j}\sigma_{t-j}^2>0$, showing that a restriction upon the behaviour of σ_t^2 is necessary. A sufficient condition to ensure Supposition (b) would be $\sigma_t^2\geq\bar{\sigma}^2>0$. By placing the uniform lower bound on σ_t^2 along with the SSE (4), this is sufficient to ensure that $N^{-1}\sum_{t=1}^{N}E(u_t^2Z_tZ_t')$ is non-singular for N sufficiently large, a condition required when the consistent covariance matrix is to be used when forming test statistics.

Now, I can prove that $\hat{\Delta}\xrightarrow{\text{a.s.}}\Delta$ and the SSE (6) is a strongly consistent estimator of the covariance matrix. In order to do this, frequent use would be made of a strong law of large numbers for martingale differences due to Chow's paper[6]. So, therefore, I can also describe the following LEMMA:

(*LEMMA*) :"If $\{x_t, \boldsymbol{F}_t, t\geq1\}$ is a martingale difference sequence such that $\sum_{t=1}^{N}E[|x_t|^{2\alpha}]/t^{1+\alpha}<\infty$, for some $\alpha\geq1$, then $N^{-1}\sum_{t=1}^{N}X_t\xrightarrow{\text{a.s.}}0$".

Having shown the strong consistency of the SSE (6) it would be possible to extend the results derived here to encompass the case of a regression model of the SSE (1) containing both lagged endogenous and lagged exogenous variables, with the latter satisfying the standard supposition $N^{-1}\Sigma x_t' x_t = Q > 0$. Allowing for stochastic regressors however would be somewhat more difficult.

If $\triangle' = (\triangle_1, \cdots\cdots, \triangle_p)$, then, the SSE (3) could be explained by the implications of the SSE (4). And then, letting $y' = (y_1, \cdots\cdots, y_N)$, $Z' = (Z_1, \cdots\cdots, Z_N)$, $u' = (u_1, \cdots\cdots, u_N)$, the SSE (5) is to be proposed.

The OLS estimate $\hat{\triangle}$ of $\triangle$ is given by $\hat{\triangle} = (Z'Z)^{-1}Z'y$. As I can show in the followings, $N^{\frac{1}{2}}(\hat{\triangle} - \triangle)$ is asymptotically normally distributed with zero mean and covariance matrix given by $\text{cov}\{N^{\frac{1}{2}}(\hat{\triangle} - \triangle)\} = V^{-1}WV^{-1}$, with $W = \lim_{N\to\infty}\left\{\frac{1}{N}\sum_{t=1}^{N}\sigma_t^2 E(Z_t Z_t')\right\} > 0$. This suggests that $\text{cov}\{N^{\frac{1}{2}}(\hat{\triangle} - \triangle)\}$ be estimated by the SSE (6).

In the above-mentioned SSE (6), $\hat{\Sigma} = \text{diag}\{\hat{u}_1^2, \cdots\cdots, \hat{u}_N^2\}$ with $\hat{u}_t$ is to be the OLS residuals from the SSE (5).

By the way, in the followings, the above-shown $N^{\frac{1}{2}}(\hat{\triangle} - \triangle)$ is to be examined. That is to say, in the following, I can show the explanatory remarks on the asymptotic normality of $N^{\frac{1}{2}}(\hat{\triangle} - \triangle)$. Since $\hat{\triangle} = (Z'Z)^{-1}Z'y$, the following form (I) is to be proposed:

(I) $N^{\frac{1}{2}}(\hat{\triangle} - \triangle) = (N^{-1}Z'Z)^{-1}N^{-\frac{1}{2}}Z'u$.

Defining c as a $p*1$ vector of constants and $\boldsymbol{F}_{t-1} = (u_{t-1}, \cdots\cdots, u_{t-p})$, $E(N^{-\frac{1}{2}}c'Z'u|\boldsymbol{F}_{t-1}) = 0$ and $E(N^{-\frac{1}{2}}c'Z'u)^2 = N^{-1}c'E(Z'uu'Z)c = c'\left\{\frac{1}{N}\sum_{t=1}^{N}\sigma_t^2 E(Z_t Z_t')\right\}c \longrightarrow c'Wc > 0$. In order to establish the asymptotic normality of $N^{-\frac{1}{2}}Z'u$, and hence $N^{\frac{1}{2}}(\hat{\triangle} - \triangle)$. I can use the following theorem. This theorem had been proposed by Professor B.M. Brown in 1971[7].

THE BROWN'S THEOREM :

"Let Ω be a martingale difference sequence with $S_N^2 = \sum_{t=1}^{N}E(\Omega_t^2)$. Then

(α) if $S_N^{-2}\sum_{t=1}^{N}E(\Omega_t^2 I\{|\Omega_t| \geq \varepsilon S_N\}) \longrightarrow 0,\ \varepsilon > 0$,

and

(β) if $S_N^{-2}\sum_{t=1}^{N}E(\Omega_t^2|\boldsymbol{F}_{t-1}) \xrightarrow{p} 1$,

then,

$S_N^{-1}\sum_{t=1}^{N}\Omega_t \xrightarrow{D} N(0, 1)$,

where $\boldsymbol{F}_{t-1}$ is the σ-sigma field generated by $\Omega_{t-1}, \cdots\cdots, \Omega_1$."

Now $c'Z'u$ has t th term $c'Z_t u_t$ which is a martingale difference. With $\Omega_t = c'Z_t u_t$, $c'Z'u = \sum_{t=1}^{N}\Omega_t$ and to determine the distribution of $N^{-\frac{1}{2}}c'Z'u$, here, I can consider $(N^{-1}S_n^2)^{-\frac{1}{2}}N^{-\frac{1}{2}}c'Z'u$.

From the proof of THEOREM I in PART III (EXAMINATIONS AND VERIFICATIONS), $N^{-1}\sum_{t=1}^{N}\{Z_t Z_t' - E(Z_t Z_t')\} \xrightarrow{\text{a.s.}} 0$ and hence I can set it in probability. Then consequently, the above-mentioned condition (β) is satisfied. Crowder showed, in his paper[8], that the condition (α) is implied by $S_N^{-(2+\delta)}\sum_{t-1}^{N}E\{|\Omega_t|^{2+\delta}\} \longrightarrow 0$ for some $\delta > 0$.

Choosing $\delta=2$, this becomes

(α') $S_N^{-4}\sum_{t=1}^{N}E(\Omega_t^4)\longrightarrow 0.$

In turn Crowder demonstrated, in his paper, that the condition (α) is implied by the proposition (A) $E(\Omega_t^4)$ bounded uniformly in t and the proposition (B) $N^{-1}S_N^4\longrightarrow\infty$.

So, the second of these, i.e., the proposition (B), is readily shown. Indeed, $N^{-1}S_N^4=N(N^{-1}S_N^2)^2\longrightarrow\infty$ since $N^{-1}S_N^2\longrightarrow c'Wc>0$.

The condition (β) in the above-mentioned Brown's theorem holds if $(N^{-1}S_N^2)^{-1}[N^{-1}\sum_{t=1}^{N}E(\Omega_t^2|F_{t-1})-S_N^2]\xrightarrow{p}0.$

As seen above $N^{-1}S_N^2=E(N^{-\frac{1}{2}}c'Z'u)^2\longrightarrow c'Wc>0$, and so here, I can concentrate on showing that the following form (II) could be proposed:

(II) $N^{-1}\sum_{t=1}^{N}E(\Omega_t^2|F_{t-1})-S_N^2\xrightarrow{p}0.$

It is straightforward to see that the left hand side of the above-mentioned form (II) could be modified as the following form i.e.,

$N^{-1}c'\{\sum_{t=1}^{N}\sigma_t^2\{Z_tZ_t'-E\{Z_tZ_t'\}\}\}c$ and this converges in probability to zero if $N^{-1}\sum_{t=1}^{T}\sigma_t^2\{Z_tZ_t'-E(Z_tZ_t')\}\leq\sigma^2N^{-1}\sum_{t=1}^{N}\{Z_tZ_t'-E\{Z_tZ_t'\}\}\xrightarrow{p}0.$

In this case, since

$E(u_{t-r}u_{t-s}u_{t-v}u_{t-w})=\sigma_{t-r}^2\sigma_{t-v}^2,\ r=s\neq v=w,$

$E(u_{t-r}u_{t-s}u_{t-v}u_{t-w})=\sigma_{t-r}^2\sigma_{t-w}^2,\ r=u\neq s=w,$

$E(u_{t-r}u_{t-s}u_{t-v}u_{t-w})=\sigma_{t-r}^2\sigma_{t-s}^2,\ r=w\neq s=v,$

$E(u_{t-r}u_{t-s}u_{t-v}u_{t-w})=\sigma_{t-r}^4,\ r=s=u=v,$

$E(u_{t-r}u_{t-s}u_{t-v}u_{t-w})=0$, otherwise, from the Supposition (c),

$E(u_{t-r}u_{t-s}u_{t-v}u_{t-w})$ is bounded and the result follows.

So, all that remains to show is that the above-mentioned proposition (A)——i.e., that $E(\Omega_t^4)$ is bounded uniformly in t—— is presented. Now, $E(\Omega_t^4)=E(c'Z_tu_t)^4=E(c'Z_t)^4E(u_t^4)$ and from the Supposition (d) for the SSE (1), $E(u_t^4)$ is bounded. And now $E(c'Z_t)^4=E(\sum_{j=1}^{p}c_jy_{t-j})^4=\sum_j^p\sum_k^p\sum_l^p\sum_m^p c_jc_kc_lc_mE(y_{t-j}y_{t-k}y_{t-l}y_{t-m})$ and what I need to show is that $E(y_{t-j}y_{t-k}y_{t-l}y_{t-m})$ is bounded.

Now, in this case, from the SSE (7), it is convenient to write $y_{t-j}=\sum_{w=j}^{\infty}a_{w-j}u_{t-w}$ and so $E(y_{t-j}y_{t-k}y_{t-l}y_{t-m})=\sum_{r=j}^{\infty}\sum_{s=k}^{\infty}\sum_{v=l}^{\infty}\sum_{w=m}^{\infty}a_{r-j}a_{s-k}a_{v-l}a_{w-m}E(u_{t-r}u_{t-s}u_{t-v}u_{t-w})$.

Since the conditions of the BROWN'S THEOREM are satisfied $N^{-\frac{1}{2}}c'Z'u\xrightarrow{D}N(0,c'Wc)$ and, using the Cramer-Wold's technique[9], it follows that $N^{-\frac{1}{2}}Z'u\xrightarrow{D}N(0,W)$.

Furthermore, from the proof of THEOREM I in PART III (EXAMINATIONS AND VERIFICATIONS), $N^{-1}Z'Z-V\xrightarrow{a.s.}0$ and so, from the above-mentioned form (I), $N^{\frac{1}{2}}(\hat{\triangle}-\triangle)\xrightarrow{D}N(0,V^{-1}WV^{-1})$ as required.

Examinations AND Verifications

First of all, I can prove the following theorem :

THEOREM I : "For the SSE (3) for which Suppositions (a)-(d) are assumed to hold, the OLS estimate $\hat{\Delta}=(Z'Z)^{-1}Z'y$ is a strongly consistent estimator of the $p*1$ vector of coefficient of the SSE (3)."

THE PROOF OF THEOREM I :

For the SSE (3), the martingale difference property of the u_t, $t=1, 2, \cdots\cdots$, and the boundedness of the variances σ_t^2 in Supposition (c) are required in order that I can use the LEMMA in PART II (METHOD). Supposition (a) guarantees that the SSE (3) can be written in the SSE (7).

In the SSE (7), the a_j decay exponentially as j increases[10]. My reason for expressing y_t in the SSE (7) becomes obvious in the proving procedures concerned with the above-mentioned THEOREM and the THEOREM II that I can show in the followings. Furthermore from the form of the independent processes I encounter, Supposition (d) is also necessary.

By the way, it is not hard to see that how the SSE (8) is to be understood.

Now, $\left(\frac{1}{N}\right)Z'Z=\frac{1}{N}\sum_{t=1}^{N}Z_tZ_t'$, the r, s th element of which is $\frac{1}{N}\sum_{t=1}^{N}y_{t-r}y_{t-s}$ and it has to be shown that $\frac{1}{N}\sum_{t=1}^{N}y_{t-r}y_{t-s}\xrightarrow{\text{a.s.}}\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}E(y_{t-r}y_{t-s})=v_{rs}$.

Considering the second term on the right hand side of the SSE (8), the r th element ($r=1, \cdots\cdots, p$) of the $p*1$ vector $\frac{1}{N}\sum_{t=1}^{N}Z_tu_t$ is $\frac{1}{N}\sum_{t=1}^{N}y_{t-r}u_t=\sum_{j=0}^{\infty}a_j\frac{1}{N}\sum_{t=1}^{N}u_{t-r-j}u_t$ using the SSE (7). Under Suppositions (c) and (d), it follows from the LEMMA in PART II (METHOD) that, for $r=1, \cdots\cdots, p$, that the SSE (10) could be proposed.

From the Supposition (a), using the SSE (7), $\frac{1}{N}\sum_{t=1}^{N}y_{t-r}y_{t-s}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_ja_k\frac{1}{N}\sum_{t=1}^{N}u_{t-r-j}u_{t-s-k}$, and for $r\geq s$ (if $r<s$ simply interchange the roles for and s in the ensuing formulae), so $\frac{1}{N}\sum_{t=1}^{N}y_{t-r}y_{t-s}=\sum_{\substack{j,k=0\\k\neq r+j-s}}^{\infty}\sum_{j,k=0}^{\infty}a_ja_k\frac{1}{N}\sum_{t=1}^{N}u_{t-r-j}u_{t-s-k}+\sum_{j=0}^{\infty}a_ja_{j+r-s}\frac{1}{N}\sum_{t=1}^{N}u^2_{t-r-j}$.

For r, $s=1, \cdots\cdots, p$; j, $k=0, \cdots\cdots, \infty$, $E(u_{t-r-j}u_{t-s-k})=0$, $(k\neq r+j-s)$, $E(u_{t-r-j}u_{t-s-k})=\sigma^2_{t-r-j}$, $(k=r+j-s)$.

However, $E(|u_iu_j|^{1+\delta})\leq E(|u_i^2u_j^2|^{1+\delta})+1$, and so Supposition (d) implies $E(|u_iu_j|^{1+\delta})\leq D_1<\infty$, where D_1 is a positive constant and i, $j=1, 2, \cdots\cdots$

And so, for $a\geq 0$, $\{[u_tu_{t-a}-E(u_tu_{t-a})], t=1, 2, \cdots\cdots\}$ is a martingale difference, and so from the LEMMA in PART II (METHOD) it follows that $\frac{1}{N}\sum_{t=1}^{N}u_{t-r-j}u_{t-s-k}\xrightarrow{\text{a.s.}}0$, if $k\neq r+j-s$, and $\frac{1}{N}\sum_{t=1}^{N}u_{t-r-j}u_{t-s-k}\xrightarrow{\text{a.s.}}\Omega_r(j)$, if $k=r+j-s$, where $\Omega_r(j)=\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\sigma^2_{t-r-j}$.

Thus $\frac{1}{N}\sum_{t=1}^{N} y_{t-r}y_{t-s} \xrightarrow{a.s.} \sum_{j=0}^{\infty} a_j a_{j+r-s}\Omega_{rs}(j)=\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}E(y_{t-r}y_{t-s})=v_{rs}$, and so the SSE (9), in this case, can be proposed.

Hence, from the SSE (8), the SSE (9) and the SSE (10), I can have $\hat{\triangle}-\triangle \xrightarrow{a.s.} 0$, as required.
……Q.E.D.

So, using the results of the above-mentioned THEOREM I, it is now possible to prove the following THEOREM II.

THEOREM II : "Under the Suppositions (a)-(d), the estimated covariance matrix of the limiting distribution of $N^{\frac{1}{2}}(\hat{\triangle}-\triangle)$ is strongly consistent."

THE PROOF OF THEOREM II :

From the SSE (6), cov $(N^{\frac{1}{2}}(\hat{\triangle}-\triangle))=\left(\frac{Z'Z}{N}\right)^{-1}\left(\frac{Z'\hat{\Sigma}Z}{N}\right)\left(\frac{Z'Z}{N}\right)^{-1}$ and from the proof of THEOREM I, $N^{-1}Z'Z-V \xrightarrow{a.s.} 0$. So, all that needs to be shown here that $V^{-1}(N^{-1}Z'(\hat{\Sigma}-\Sigma)Z)V^{-1} \xrightarrow{a.s.} 0$.

For $r, s=1, \cdots\cdots, p$, the r, s *th* element of the first term on the right hand side of the SSE(11)is $-\frac{2}{N}\sum_{t=1}^{N}\sum_{k=1}^{p} y_{t-r}y_{t-k}y_{t-s}u_t(\hat{\triangle}_k-\triangle_k)$.

Now consider the following:

$\frac{1}{N}Z'(\hat{\Sigma}-\Sigma)Z=\frac{1}{N}\sum_{t=1}^{N}Z_t(\hat{u}_t^2-u_t^2)Z_t'=\frac{1}{N}\sum_{t=1}^{N}Z_t(\hat{u}_t+u_t)(\hat{u}_t-u_t)Z_t'$, then it is not hard to see that $(\hat{u}_t+u_t)=2u_t-Z_t'(\hat{\triangle}-\triangle)$, $(\hat{u}_t-u_t)=-Z'(\hat{\triangle}-\triangle)$, so that in this case, the SSE (11)can be proposed.

THEOREM I demonstrates that $\hat{\triangle}_k \xrightarrow{a.s.} \triangle_k (k=1, \cdots\cdots, p)$.

Now, consider $\frac{1}{N}\sum_{t=1}^{N} y_{t-r}y_{t-k}y_{t-s}u_t$. I can express y_{t-l}, $l=k, r, s$, as a linear process of the SSE (7) and use an argument similar to that used to investigate the almost sure convergence of $N^{-1}Z'Z$. By doing this it is seen that $\frac{1}{N}\sum_{t=1}^{N} y_{t-r}y_{t-k}y_{t-s}u_t=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty} a_a a_b a_c \frac{1}{N} * \sum_{t=1}^{N} u_{t-r-a}u_{t-k-b}u_{t-s-c}u_t$.

And by defining $W_t(a. b, c, k, s)=u_{t-r-a}u_{t-k-b}u_{t-s-c}u_t$, where $a, b, c=0, \cdots\cdots, \infty$; $k, r, s=1, \cdots\cdots, p$, the $\{w_t(a, b, c, r, k, s), t=1, 2, \cdots\cdots\}$ are a sequence of martingale differences.

Thus from Supposition (d) and the LEMMA in PART II (METHOD), it is clear that $\frac{1}{N}\sum_{t=1}^{N} w_t(a, b, c, r, k, s) \xrightarrow{a.s.} E\{W_t(a, b, c, r, k, s)\}=0$. Hence $\frac{1}{N}\sum_{t=1}^{N} y_{t-r}y_{t-k}y_{t-s}u_t \xrightarrow{a.s.} 0$.

And therefore, in this case, the SSE(12)could be proposed.

A similar argument to that for the first term applies to the second term on the right hand side of the SSE(11), so that the SSE(13)can be defined.

Therefore, from the SSE (9) and the SSE(14), it follows $N^{-1}(Z'Z)^{-1}(Z'\hat{\Sigma}Z)(Z'Z)^{-1}$ is a strong consistent estimator of the covariance matrix of the limiting distribution of $N^{\frac{1}{2}}(\hat{\triangle}$

$-\triangle$). And from the SSE(11), the SSE(12)and the SSE(13), the SSE(14)can be proposed.

……Q.E.D.

Results

—— Simulation Experiments by operating (Cu(IBM) AMDAHL S. 470-V/8) computor machine and their Results ——

Random numbers for u_t were generated with a thoroughly tested algorithm due to Brent[11]; the initial value y_0 was set to zero and 500 observations on y_t were constructed with only the final 110 used, in order to eliminate any influence of the initial condition. 1, 000 replications were made. The following table records the actual variance of the OLS estimator (0) and the ratio of the variance estimated by the SSE (6) to this actual variance (ψ), the classification being by the three parameters $\triangle$, N, and Ω (coefficient value, sample size, and relative variances). And all values for O in the table are the actual variances multiplied by 10^2.

No evidence was presented by White et al. of the closeness of their estimators to the true convariance matrix for the sample sizes of interest for econometric research based on time series. This is probably explained by their emphases on cross-section works where the number of observations is likely to be very large. However, as it seems likely that their adjustments to the OLS variance will also be utilized in the time series context, a brief examination of the adequacy of the approximation is warranted. To this end the SSE(15) was adopted. In this SSE(15), for $N_1 \leq N$, $\sigma_t^2=1$, $t=1, \cdots\cdots, N_1$; $\sigma_t^2=\Omega$, $t=N_1+1, \cdots\cdots, N$; that is the variance is assumed to shift within the sample period. In the basic experiment $N_t=\frac{1}{2}N$ and three values of $\triangle$(0.4, 0.6, 0.8), three values of N(50, 70, 110) and three values of Ω(2, 10, 100) are chosen. These would seem to provide a reasonable description of the types of parameter values that might characterize and actual investigation.

TABLE

N \ O, Ψ \ $\triangle$ \ Ω		$\Omega=2$			$\Omega=10$			$\Omega=100$		
		$\triangle=0.4$	$\triangle=0.6$	$\triangle=0.8$	$\triangle=0.4$	$\triangle=0.6$	$\triangle=0.8$	$\triangle=0.4$	$\triangle=0.6$	$\triangle=0.8$
N=50	O	1.95	2.01	0.70	3.03	2.89	1.52	3.82	3.00	2.00
	Ψ	0.89	0.85	0.90	0.88	0.90	0.70	0.85	0.81	0.65
N=70	O	1.35	0.95	0.62	2.06	1.60	0.79	2.52	2.01	0.92
	Ψ	0.96	0.92	0.75	0.90	0.90	0.74	0.86	0.82	1.05
N=110	O	0.85	0.49	0.36	1.30	0.99	0.40	1.73	0.95	0.60
	Ψ	0.82	0.80	0.92	0.78	0.82	0.72	0.85	0.82	1.01

Conclusion

These results that were shown in PART IV (RESULTS —— i.e SIMULATION EXPERIMENTS BY OPERATING CU (IBM) AMDAHL S. 470-V/8 COMPUTOR

MACHINE AND THEIR RESULTS) are very interesting. They reveal that, for moderate amounts of heteroscedasticity and an autoregressive parameter which is not too large, White et al. adjusted variance is a good approximation to the actual one. This conclusion holds even in small samples. However, as the degree of heteroscedasticity becomes very large and as $\triangle$ tends to unity, there can be very large discrepancies between estimated and actual variances. A relative variance of 100 is probably somewhat unrealistic in a step function and the values for $\Omega=10$ are perhaps a better guide to actual applied data. The variation observed in estimator performance as $\triangle$ increases may not be surprising, as a similar result[12] for OLS occurs when disturbances are homoscedastic[13].

Finally, it is worth mentioning that changing N_1 from $\frac{N}{2}$ to $\frac{N}{4}$ resulted in only minor changes to the results.

References

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3) Professor White suggested such an extension in an unpublished memorandum and provided conditions for consistency and asymptotic normality of OLS. My set of conditions given later is more specialied but perhaps easier to interpret.
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10) If exogenous variables are present I would write the following : $y_t=\sum_{j=0}^{\infty}\gamma_j x_{t-j}+\sum_{j=0}^{\infty}a_j u_{t-j}$.. The following proofs are then easily generalized.
11) Brent, R.P.: A Gaussian Pseudo Random Number Generator, Communications of the ACM., **17**, 704~706 (1974)
12) Orcutt, G.H., and H.S. Winokur: First Order Autoregression: Estimation and Prediction, Econometrica, **69**, 1~14 (1976)
13) The ratio of the SSE (6) to the theoretical asymptotic variance was close to one. Thus the discrepancy between the actual and estimated variances can be interpreted as a failure of the asymptotic formula to accurately reflect the actual variance when $\triangle$ is close to unity and the sample size is not very large.

(An Additional Remarks):

This paper is an extensively and revised one of the study which I had reported at The

International Meeting of The INTERNATIONAL SOCIETY OF OPERATIONS RESEARCHES SCIENCES that had been held at Washington D.C., US, in August 6-12, 1984. And I had done the computor-simulation experiments that had been explained in this paper by operating CU (IBM) AMDAHL S. 470-V/8 computor machine in Columbia University, New York, US. in August 6-9, 1984.

要　約

大型電算機{CU (IBM、AMDAHL)、システム 470-V/8}処理にもとづく
SD モデルの誤差分散の非均質性に関する吟味と検出

この小論で筆者は、SD モデルにおける整合型共分散行列の推定子と誤差分散の非均質性に関する直接法テストをめぐる問題をとり扱い、SD モデルの誤差分散の非均質性に関するホワイトやクラグに代表される従来の各論証結果に対する批判的吟味を行ない、さらに、個々の SD モデルに採択されている外生変数がラグつきの内生変数に代位可能となるとき、実際のシステム展開に事実上有効なものとなるこれら論証結果のうちの各ケースを、コンピューターシミュレーションの作業をとおしてそれぞれ検出した。

（付記）：なお、この小論は、昨夏（1984）8 月、米国ワシントン D.C.で開催された国際 OR 学会での研究報告論文を修正加筆したものである。

ちなみに、うえにのべたコンピューターシュミレーションの作業は、昨夏 8 月初旬、各データベースのバッテリーとエントリィのためのセッティングをすべて日本ですませたファイルを米国コロンビア大学大型電子計算センターへ搬入し、{CU. (IBM) AMDAHL S. 470-V/8}のマシンに入力し、筆者自身がおこなったものである。

なお、紙幅の都合でプログラムリストはすべてこの小論での掲載を割愛した。