A Study on Scheffman's Proposition

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Introduction

First of all, consider a decision problem of the form

(1) $$\max_{\alpha \in A} \eta(\alpha) = E\varphi(X + \alpha Y),$$

where X is a random variable representing a decision maker's current prospect, Y is a random variable representing a possible venture and φ is his utility function for future wealth.

The current prospect reflects the decision maker's possible values of future wealth if he proceeds with his present plans, commitments, business undertakings, investments, etc.. The venture Y is a prospective security purchase or sale, business deal, insurance policy, or other project that, if undertaken, may influence future wealth. If α is the amount of he venture undertaken, $X + \alpha Y$ becomes the decision maker's prospect. Direct applicability of this simple model is limited by several of the assumptions. It is a traditional two-period model with just one prospective venture. Leland[1] has shown that familiar conditions for a maximum apply if α, Y are interpreted as vectors. Fama[2] has shown that, for some problems, the two-period model can be embedded in a multi-period model. For some business ventures, the linearity and additivity assumptions may not be appropriate ; but it may sometimes be possible to approximate a single nonadditive or nonlinear venture by several appropriately restricted linear and additive ventures. A represents possible amounts of the venture.

A is determined by the circumstances of the particular venture under consideration. A purchase of commom stock could be any nonnegative integral number of shares up to the limit of the decision maker's resources. Stock options or commodity futures could be bought or sold so α could be possitive or negative. Joining a partnership might require a specified investment so A would be a single point. It is convenient to initially assume that α might be any real number and subsequently consider restrictions that might be imposed in particular applications.[3] Unless otherwise noted, A is assumed equal to R, the real line.

If φ is differentiable and concave (risk aversion) and if $E\ \varphi(X + \alpha Y)$, $E\ Y\ \varphi'(X + \alpha Y)$ are finite then η is differentiable[4] and

(2) $$\eta'(\alpha) = EY\varphi'(X + \alpha Y).$$

φ strictly concave implies η strictly concave in which case the optimal (maximizing) value of α, say $\hat{\alpha}$, is unique if it exsts. $\hat{\alpha}$ exists if and only if $P(Y>0)>0$ and $P(Y<0)>0$ (neither Y nor $-Y$ is a sure thing)[5].

One is interested in relating $\hat{\alpha}$ to properties of the initial prospect and the venture that may sometimes be determinable in practical situations. With strict concavity, hereinafter assumed, $\hat{\alpha}$ uniquely solves $\eta'(\alpha)=0$. $\eta(\alpha)$ has the form of an inverted U so one way to investigate the general location of $\hat{\alpha}$ is to try to determine the sign of $\eta'(\alpha)$ for intersting values of α. If $\eta'(\bar{\alpha})>0$ for a chosen $\bar{\alpha}$ then $\hat{\alpha}>\bar{\alpha}$ since $\eta(\alpha)$ must level off to the right of $\bar{\alpha}$. Thus

(3) $\quad \hat{\alpha} \gtreqless \bar{\alpha} \Longleftrightarrow \eta'(\bar{\alpha}) \gtreqless 0.$

Thus determining the sign of $\eta'(0)$, marginal expected utility at the origin, indicates the sign of $\hat{\alpha}$. If the decision maker can either buy or sell short, a negative $\hat{\alpha}$ would indicate the latter. In cases where only positive amounts of the venture are feasible, negative $\hat{\alpha}$ indicates that he would retain his current prospect. Recalling (2),

(4) $\quad \eta'(0)=EY\varphi'(X)=(EY)(E\varphi'(X))+Cov(Y, \varphi'(X)).$

To determine the common sign of $\eta'(0)$ and $\hat{\alpha}$, note that a decision maker's normal preference for higher income implies that φ' and therefore $E\varphi'$ is positive. Thus, if EY and $Cov(Y, \varphi'(X))$ agree in sign, $\eta'(0)$ and $\hat{\alpha}$ will also have this sign. IF EY and $Cov(Y, \varphi'(X))$ differ in sign one has to know further particulars of the utility function and the joint distribution of X, Y to determine the sign of $\hat{\alpha}$.

If X and Y are stochastically independent, then $Cov(Y, \varphi'(X))=0$ and $\hat{\alpha}$ agrees in sign with EY.

The main purpose of this paper is to indicate several conditions sufficient to determine the sign of $Cov(Y, \varphi'(X))$ when the initial prospect and venture are not independent. The conditions and some logical relations among them are given in the following section. And some hypothetical applications are cited in the following section of "Results and Verifications".

Methods

—— Sufficient Conditions for Positive Covariance ——

Under risk aversion, φ' is decreasing so Condition (a) below makes $Cov(Y, \varphi'(X))$ positive. Each of the other conditions is shown to imply Condition (a) and relations among the other conditions are explored. As is customary if F, G are probability distribution functions, $F<G$ is defined to mean $F(X)\leq G(X) \forall x \varepsilon R$ and $F \neq G$.

THEOREM I "Let X, Y be nondegenerate random variables with finite means and variances and with distribution functions F_x, F_y. The following implications hold among the conditions hold among the conditions listed below : (b) $\Rightarrow$ (a), (c) $\Rightarrow$ (b), (d) $\Rightarrow$ (b), (e) $\Rightarrow$ (c), (f) $\Rightarrow$ (b), (g) $\Rightarrow$ (a), (h) $\Rightarrow$ (g), (h) $\Rightarrow$ (e). Statements about conditional expectations and distributions in (b) through (e) are to be understood to hold $a.s.\ F_x$.

Condition (a) : Y is positively correlated with any strictly decreasing function of X with finite second moment.

Condition (b) : $\exists\, \tilde{x} \ni [x < \tilde{x} \Rightarrow E(Y \mid X=x) > E_y],\ [x > \tilde{x} \Rightarrow E(Y \mid X=x) < EY]$.

Condition (c) : $E(Y \mid X=x)$ is a strictly decreasing function of x.

Condition (d) : $\exists\, \tilde{x} \ni [FY \mid_{X=x} X = X \gtreqless Fy \Longleftrightarrow x \lesseqgtr \tilde{x}]$.

Condition (e) : $\forall\, \tilde{x},\ \tilde{x} \in$ support F_X with $\tilde{x} < \tilde{x}$, $F_{Y|X} X = \tilde{x} < F_{Y|X} X = \tilde{x} < F_{Y|X} = \tilde{x}$.

Condition (f) : $\exists\, \tilde{x} \ni [Y \gtreqless EY \Longleftrightarrow X \gtreqless EY \Longleftrightarrow X \lesseqgtr \tilde{x}]$

Condition (g) : $Y = F(W, V)$ and $X = g(W, Z)$ where W is a nontrivial random variable ; V and Z are ramdom mappings ; W, V, Z are independent ; $f(\cdot, \cdot)$ is strictly increasing in its first argument ; $g(\cdot, \cdot)$ is strictly decreasing in its first argument.

Condition (h) : $Y = f(W, V)$ and $X = g(W)$ where W, V, f are as in the Condition (g) and g is strictly decreasing"

PROOF :

(b) $\Longrightarrow$ (a)

Let $\gamma : R \rightarrow R$ be strictly decreasing and $Y^* = Y - EY$. Then $Cov(Y, \gamma(x)) = EY^*\gamma(X) = \int_{X<\tilde{X}} E(Y^* \mid X=x)\gamma(x)dF_X + \int_{X \geq \tilde{X}} E(Y^* \mid X=x)\gamma(x)dF_X > \gamma(\tilde{x}) \int E(Y^* \mid X = x)dF_X = \gamma(\tilde{x})EY^* = 0$

(c) $\Longrightarrow$ (b)

Choose a version of $E(Y \mid X=x)$ that is strictly decreasing. Let $\tilde{x} = sup\{x : E(Y \mid X=x) > EY\}$.

(d) $\Longrightarrow$ (b)

The following Lemma is proved in Tesfatsion's paper[6] and modifies an earlier Lemma by Hanoch and Levy.

LEMMA : If F, G are distribution functions and $O : R \rightarrow R$ is continuous, nondercreasing and $\int \theta dF < \infty$, $\int \theta dG < \infty$; then $\int \theta dF - \int \theta dG = \int (G-F) d\theta$. Suppose $x > \tilde{x}$. Then $F_{Y|X=x} > F_Y$ $a.s.$ and $E(Y|X=x) - EY \overset{as}{=} \int y dF_{Y|X=x} - \int y dF_Y = \int (F_Y(y) - F_{Y|X=x}(y)) dy < 0$.

(e) $\Longrightarrow$ (c)

Similarly, use of the above Lemma. That (e) $\Rightarrow$ (c) is essentially the same as a proposition of D. T. Scheffman[7].

(f) $\Longrightarrow$ (b)

$E(Y|X=x) \overset{as}{=} \int y F_{Y|X=x}$ and for $x \langle (resp \rangle) \tilde{x}$, $y > (<) EY$.

(g) $\Longrightarrow$ (a)

Let $\gamma : R \rightarrow R$ be any strictly decreasing function. Define $h(W, Z) = \gamma(g(W, Z))$. Clearly h is strictly increasing in its first arguments. Define $\bar{f}(w) = EF(w, V)$ and $\bar{h}(w) = Eh(w, Z)$. $\bar{f}$ and $\bar{h}$ are strictly increasing. Without loss of generality let $EY = 0$. Define $w_o = inf\{w \mid \bar{f}(w) > 0\}$. Then $Cov(Y, \gamma(X)) = E(Y, \gamma(X)) = E(f(w, V) h(W, Z)) = \int \bar{f}(w) \bar{h}(w) dF_W(w) = \int_{W \geqq W_o} \bar{f}(w) \bar{h}(w) dF_W(w) + \int_{W < W_o} \bar{f}(w) \bar{h}(w) dF_W(w) > \bar{h}(w^o) \int \bar{f}(w) dF_W(w) = \bar{h}(w_o) EY = 0$.

This result generalizes another of Scheffman's Lemma[8].

(h) $\Longrightarrow$ (g)

Obvious since (h) may be regarded as the special case of (g) in which is constant.

(h) $\Longrightarrow$ (e)

Since g is strictly decreasing, g^{-1} exists and d decreases strictly. Write $W = g^{-1}(X)$ and $Y = f(g^{-1}(X), V) = h(X, V)$ where h is strictly decreasing in its first argument. For any $\tilde{x} < \hat{x}$, $E(Y|X=\tilde{x}) \overset{as}{=} Eh(\tilde{x}, V) > Eh(\hat{x}, V) \overset{as}{=} E(Y|X=\hat{x})$ since $h(\tilde{x}, v) > h(\hat{x}, v)$ for all v. For any y, $F_{Y|X=\tilde{x}}(y) \overset{as}{=} P_V(\{v : h(\tilde{x}, v) \leq y\}) \leq P_V(\{v : h(\hat{x}, v) \leq y\}) \overset{a.s}{=} F_{Y|\hat{x}}(y)$ since $\{v : h(\tilde{x}, v) \leq y\} \subset \{v : h(\hat{x}, v) \leq y\}$. The distributions cannot be equal since it was shown that $E(Y|X=\tilde{x}) > E(Y|X=\hat{x}) a.s.$.

Summarising, and taking account of transitivity of $\Rightarrow$, we can have COROLLARY like the below-shown. That is to say, For the conditions of THEOREM I, (h) $\Rightarrow$ (a), (c) $\Rightarrow$ (b), (a), (d) $\Rightarrow$ (b), (a), (e) $\Rightarrow$ (c), (b), (a), (f) $\Rightarrow$ (b), (a), (g) $\Rightarrow$ (a), (h) $\Rightarrow$ (g), (e), (c), (b), (a).

Whether other implications might exist is a natural question answered by THEOREM II "Consider random variables X, Y and Condition (a) through (h) as in

THEOREM I. The implications listed in the COROLLARY like the above-mentioned are the only valid implications among these conditions."

PROOF :

(h) $\not\Rightarrow$ (f), (h) $\not\Rightarrow$ (d)

ⓐ : Denote the probability space on which the random variables are defined by (Ω, F, P). Let $\Omega=\{1, 2, 3, 4\}$ with respective probabilities 0.3, 0.2, 0.2, 0.3. Let $W(1)=W(3)=0$, $W(2)=W(4)=1$, $V(3)=V(4)=0$, $V(1)=V(2)=1$. Let $X=-W$ and $Y=W+2V$. A little arithmetic verifies that (h) holds but neither (f) nor (d).

(g) implies only (a)

ⓑ : Let W, V, Z of Condition (g) each take the value -1 with probability 0.5 and 1 with probability 0.5 and be independent. Let $Y=V+\varepsilon W$, $X=Z-\varepsilon W$ where $0<\varepsilon<1$. (g) is satisfied. Using independence of W, V, Z, a simple calculation yields $E(Y\,|\,X=-1-\varepsilon)=\varepsilon$, $E(Y\,|\,X=-1+\varepsilon)=-\varepsilon$, $E(Y\,|\,X=1-\varepsilon)=\varepsilon$, $E(Y\,|\,X=1+\varepsilon)=-\varepsilon$, which violate (b). (g) cannot imply (c) since (c) implies (b). Similary (g) cannot imply (d), (e), (f) or (h) since each of these implies (b), (f) $\neq$ (h), (g),

(e), (d), or (c).

ⓒ : Let $\Omega=\{1, 2, 3, 4\}$ with $P\{w\}=0.25\forall w$; $X(w)=w\forall w$; $Y(1)=11$, $Y(2)=Y(4)=0$, $Y(3)=1$. Then $EY=3$ and $\tilde{x}=1.5$ makes (f) satisfied. However, (c), (d), (e), (g), (h) are violated. To see that (g) and (h) are not satisfied note that any Z independent of Y would have to be a constant as would any V independent of X. Thus to satisfy (g) or (h) there would have a monotonically related to both X and Y. But this is impossible since X is monotonic on Ω and Y is not.

(e) $\not\Rightarrow$ (h), (g), (f), or (d).

ⓓ : Let $\Omega=\{1, 2, 3, 4, 5\}$ with $P(w)=0.2\forall \omega$; $Y(w)=\omega$, $X(1)=X(3)=1$, $X(2)=X(4)=0$, $X(5)=-1$. (e) q satisfied. By an argument similar to that in (c), (g) is not satisfied. Since (h) $\Rightarrow$ (g), (e) $\not\Rightarrow$ (g) it follows that (e) $\not\Rightarrow$ (h). Since (h) $\Rightarrow$ (g), (h) $\not\Rightarrow$ (d) it follows that (e) $\not\Rightarrow$ (d) ; a similar argument shows (h) $\not\Rightarrow$ (d).

(d) $\not\Rightarrow$ (h), (g), (d), (e), or (c).

ⓔ : Let $\Omega=\{1, 2, 3, 4, 5\}$ with probabilities 0.67, 0.3, 0.3, 0.67. Let $X(0)=0$, $X(2)=X(3)=1$, $X(4)=2$ and let $Y(1)=Y(3)=1$, $Y(2)=Y(4)=0$. (d) holds with $\tilde{x}=1$. (g), and therefore (h), does not hold. Tentatively suppose (g) holds. Any V independent of X must be constant so we may write $Y=f(W)$, $W=f^{-1}(Y)$, and $X=g(f^{-1}(Y), Z)=h(Y, Z)$ where h is strictly decreasing in its first argument. Let $Z(1)=\mu$, $Z(3)=\nu$. Then Z independent of Y requires that $Z(2)=\nu$, $Z(4)=\mu$. But then $h(1, \nu)=h(0, \nu)=1$ which contradicts the fact that

h is strictly decreasing in its first argument.

ⓕ : Let $\Omega=\{1, 2, \cdots\cdots, 7\}$ with $P\{w\}=0.08333$ for $\omega=1 \cdots 4$, $P\{5\}=0.17$, $P\{6\}=P\{7\}=0.25$. Let $X(1)=X(5)=X(6)=0$, $X(2)=1$, $X(3)=X(4)=2$, $X(7)=-1$; and $Y(1)=Y(4)=4$, $Y(2)=Y(3)=Y(5)=0$, $Y(6)=Y(7)=5$. Then (d) holds with $\tilde{x}=o$; but (f), (e), (c) do not hold.

(c) implies only (b) and (a).

ⓖLet $\Omega=\{1, 2, 3,\}$, $P(\omega)=0.3 \forall \omega$. Let $X(1)=0$, $X(2)=X(3)=1$. Let $Y(1)==1$, $Y(2)=2$, $Y(3)=-2$. Then (c) holds but not (e). Since (e) $\Rightarrow$ (c) and (e) $\not\Rightarrow$ (h), (g), (f), or (d) ; it follows that (c) $\not\Rightarrow$ (h), (g), (f), or (d).

(b) implies only (a).

Since (d) $\Rightarrow$ (b) and (d) $\not\Rightarrow$ (h), (g), (f), (e), or (c) , (b) does not imply any of the latter. Since (c) $\Rightarrow$ (b) and (c) $\not\Rightarrow$ (d), (b) $\Rightarrow$ (d).

(a) implies none of the others.

Since (g) $\Rightarrow$ (a) but (g) $\not\Rightarrow$ (b) (a) $\not\Rightarrow$ (b). Since each of the others implies (b), (a) could not imply any without implying (b).

Quite a few propositions closely related to those of THEOREM I may be obtained by reversing or weakening appropriate inequalities and monotonicities in both assumptions and conclusions. For example, (b′) $\Rightarrow$ (a′) and (c*) $\Rightarrow$ (a*) where (b′) $\exists \tilde{x} \ni [x>\tilde{x} \Rightarrow E(Y|X=x)>EY]$, $[x<\tilde{x} \Rightarrow E(Y|X=x)<EY]$. (a′) Y is negatively correlated with any strictly decreasing function of X that has finite second moment. (c*) $E(Y|X=x)$ is a nonincreasing function of x. (d*) Y is not negatively correlated with any nonincreasing function of X with finite second moment.

Other possible modifications seem reasonably clear and too numerous to try to list.

Concluding Remarks

——The Case of Insurance——

Condition (g), called nogative S-Correlation by Scheffman, has been found useful by Samuelson[9] and Scheffman[10] in establishing several theorems on diversification of investments. Some illustrative applications of other conditions follow. The general assumptions of the section of "Methods" (e.g., the strict concavity of the utility function φ) will be assumed to hold throughout.

A decision maker stands to lose an amount $w>o$ if the event A occurs. In exchange for a premium c, he is offered an insurance policy that will cover this contingent loss. Viewed as a venture the policy can be written $Y=wI_A-c$ where I_A is the indicator function of the event A. Suppose he can also elect partial coverage at a proportionally reduced premium, i.e., he can elect to pay a premium αc, $0 \leq \alpha \leq 1$, and be reimbursed αw if the loss occurs.

Let Z represent his current prospect other than this possibility of lose. His expected utility as a function of the chosen coverage is then.

$$(5) \qquad \eta(\alpha)=E\varphi(Z=wI_A+\alpha(wI_A-c))=E\varphi(X+\alpha Y)$$

with $X=Z-wI_A$, $Y=wI_A-c$. Assuming Z is independent of A, Condition (g) of the section of "Methods" is satisfied. To observe the circumstances under which some coverage will be taken, note

(6) $$\eta'(0)=EY\ E\varphi'(X)+\mathrm{Cov}(Y,\varphi'(X)).$$

Since (THEOREM I) (g) ⇒ (a), we know that the covariance is positive. Examining the first term on the right,

(7) $$EY\ E\varphi'(X)=(wP_A-c)E\varphi'(X)$$

one observes that $E\varphi'$ is always positive and (wP_A-c) is the subjective actuarial value[11] of the policy. From (6), (7) and (3),

(8) $$\hat{\alpha}\gtreqless 0 \Leftrightarrow \eta'(0)\gtreqless 0 \Leftrightarrow (wP_A-c)\gtreqless\frac{-\mathrm{Cov}(Y,\varphi'(x))}{E\varphi'(X)}.$$

Since the ratio on the right is known to be negative, it is clear that some coverage will be chosen if the subjective actuarial value is nonnegative or even somewhat negative, so long as

(9) $$c<wP_A+\frac{\mathrm{Cov}(Y,\varphi'(X))}{\mathrm{E}\varphi'(X)}.$$

Calculation of this upper limit on the premium would, of course, require detailed knowledge of decision maker's utility and subjective probability.

One may also be interested in the circumstances under which full coverage will be taken. By (3) this depends on (1). In this case,

(10) $$\eta'(1)=EY\varphi'(X+Y)=E(wI_A-c)\varphi'(Z-c)=E(wI_A-c)E\varphi'(Z-c),$$

the final equality following from the independent of Z and A. So,

(11) $$\hat{\alpha}\gtreqless 1 \Leftrightarrow wI_A-c\gtreqless 0.$$

Thus full coverage will be desired if the policy is offered at exactly subjective actuarial value and less (more) than full coverage if the policy offers less (more) than subjective actuarial value.

The results readily extend to more general kinds of coverage. Let W be any pattern of potential loss and c a premium covering such a loss. Then $X=Z-W$, $Y=W-c$.

Condition (g) is still satisfied (assuming Z, W independent) and

$$\text{(12)} \qquad \alpha \gtreqless o \Leftrightarrow (EW-c) \gtreqless \frac{-\mathrm{Cov}(W, \varphi'(X))}{E\varphi'(X)}, \ \hat{\alpha} \gtreqless 1 \Leftrightarrow EW-c \gtreqless o,$$

where $EW-c$ is the subjective actuarial value and $\mathrm{Cov}(W, \varphi'(X))$ is known to be positive.

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8) Sheffman, D. T.: ibid. 82
9) Samuelson, P.A.: General Proof that Diversification Pays, J. Fin. & Quantitative Anal., 3, 1～13 (1967)
10) Scheffman, D.T.: ibid. 36

Notes

1) The subjective actuarial value could be different for the decision maker and the insurance company if they have different estimates of P_A, or if an uninsured porperty loss by the decision maker would involve secondarry losses—loss of customers, borrowing on unfavorable terms, etc. . In the latter case, the actual claim would be less than w.

(邦 文 要 約)

(シェフマンの命題に関する一考察)

この小論では，シェフマンが1973年に期待投機心理に関して提起した $\mathrm{Cov}(Y, \varphi'(X))=0$ の命題をめぐって種々議論されている諸論点のうちの重要なひとつである符号条件の確定に関する主題だけに焦点をしぼって，その確定のための個々の充分条件をそれぞれクリアし，その定化式と数量化理論にもとづく吟味を作業した。とくに，それら充分条件のひとつひとつ相互の数理的，論理的な諸関連をできる限り精密に検出し，その検出効果のすべてについて，保険という日常的なケースを材料に用いたコンピューターシミュレーションをとおしてチェックした。なお，そのコンピューターシミュレーションのための使用機種は，IBM S/360 (M 158, 3330) である。

(付記)：この小論は，私の Ph. D. 論文の第3部のうちの第3主題「期待投機心理の計量経済分析」

を修正加筆して新規にかき改めたものであり，来る 1986 年 1 月のミネソタ大学経済学研究所での国際シンポジウムでの研究報告のアサインメントをすでにうけた論文である．